Pergamon

Appl. Math. Lett. Vol. 7, No. 2, pp. 45–48, 1994

0893-9659/94 $6.00 + 0.00

0893-9659(94)E0010-9

Stabilization of the Angular Velocity of the Rigid Body About the Middle Axis

R. OUTBIB AND J. C. VIVALDA
INRIA Lorraine—Projet CONGE
and URA CNRS 399—MMAS Université de METZ
Technopôle METZ 2000—CESCOM
Metz, France

(Received and accepted October 1993)

Abstract—In this paper, we investigate the problem of the stabilization of the angular velocity of the rigid body with one control torque applied on the minor axis. The trajectories of this system evolve on cylinders. We study the stability when restricted to these invariant manifolds and we give a feedback law which stablizes the rigid body about the middle axis.

Keywords—Nonlinear systems, Stabilization, Rigid Body.

1. INTRODUCTION

There has been much interest in the problem of stabilizing the Euler angular velocity of the rigid body with $n \leq 2$ torques. Bloch and Marsden consider in [1] the problem of the stabilization of a rigid body around the middle axis with one control torque applied about its minor axis. This problem has also been considered differently by Aeyels in [2]. Notice that this problem is different from one which is concerned with the stabilizability of the *origin* of the Euler equations and which has been studied in [2–5]. In this communication, we analyze more closely this question and prove results about the stability. To be more precise, the rigid body equations with a single torque about the minor axis are:

$$\begin{aligned} \dot{\omega}_1 &= a_1\omega_2\omega_3, \\ \dot{\omega}_2 &= a_2\omega_3\omega_1, \\ \dot{\omega}_3 &= a_3\omega_1\omega_2 + u, \end{aligned} \tag{1}$$

where $a_1 = \dfrac{I_2 - I_3}{I_1}$, $a_2 = \dfrac{I_3 - I_1}{I_2}$ and $a_3 = \dfrac{I_1 - I_2}{I_3}$, I_1, I_2, and I_3 being the principal moments of inertia (we assume $I_1 > I_2 > I_3$). The free rigid body equation (system (1) with $u = 0$) has relative equilibria when

$$\begin{aligned} (\omega_1, \omega_2, \omega_3) &= (\Omega, 0, 0), \\ (\omega_1, \omega_2, \omega_3) &= (0, 0, \Omega), \end{aligned}$$

and when

$$(\omega_1, \omega_2, \omega_3) = (0, \Omega, 0).$$

The first two cases, which correspond to rotation about the major or minor axis, are well-known to be stable, while the last case, rotation about the intermediate axis, is unstable.

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We study the stability when restricted to the invariant manifolds: more precisely, it is clear that the trajectories of system (1) evolve on cylinders which are the level surfaces of the function $\omega \mapsto -a_2\omega_1^2 + a_1\omega_2^2$; these trajectories satisfy on the cylinder the following equations:

$$\begin{aligned} \dot{r} &= 0, \\ \dot{\theta} &= -\sqrt{-a_1a_2}\,\omega_3, \\ \dot{\omega}_3 &= \frac{a_3}{\sqrt{-a_1a_2}}\, r^2 \cos\theta \sin\theta + u, \end{aligned} \tag{2}$$

where $r = \sqrt{-a_2\omega_1^2 + a_1\omega_2^2}\,\omega_1 = (-a_2)^{-1/2} r\cos\theta$ and $\omega_2 = (a_1)^{-1/2} r \sin\theta$. The intersection set $\mathcal{S}$ between the middle axis and the cylinder is constituted by two points A and B, the question is: can we find a feedback law in such a manner that point A is an asymptotically stable equilibrium point for closed-loop system (1) when restricted to the cylinder.

2. MAIN RESULT

We are looking for feedbacks for which the singular points of the closed-loop system (2) are in $\mathcal{S}$ and this for each cylinder $-a_2\omega_1^2 + a_1\omega_2^2 = c$. Since we consider only analytic feedbacks, the set of singular points is exactly equal to $\mathcal{S}$.

PROPOSITION 1. *The closed-loop system*

$$\begin{aligned} \dot{\omega}_1 &= a_1\omega_2\omega_3, \\ \dot{\omega}_2 &= a_2\omega_1\omega_3, \\ \dot{\omega}_3 &= -\omega_3 - a_2\omega_1(-a_2\omega_1^2 + a_1\omega_2^2), \end{aligned} \tag{3}$$

defined from system (1) by the feedback:

$$u(\omega) = -a_3\omega_1\omega_2 - \omega_3 - a_2\omega_1(-a_2\omega_1^2 + a_1\omega_2^2)$$

is stable about the origin.

PROOF. Consider the function V defined by:

$$V(\omega) = \frac{1}{2}\left(\omega_3^2 + (-a_2\omega_1^2 + a_1\omega_2^2)\left(-a_2\omega_1^2 + a_1\omega_2^2 + \omega_2 + \frac{1}{a_1}\right)\right).$$

Clearly V is positive definite; if we compute $\dot{V}$, the derivative of V along the trajectories of the closed-loop system, we get:

$$\dot{V}(\omega) = -\omega_3^2,$$

which is nonpositive. ∎

Following the Lasalle's principle [6], we can say that the trajectories of the closed-loop system tend towards the largest invariant set contained in the locus $\{\omega \mid \dot{V}(\omega) = 0\}$. Now along the plane of equation $\omega_3 = 0$, the components of the vector field which defines system (3) are:

$$\begin{pmatrix} 0 \\ 0 \\ -a_2\omega_1(-a_2\omega_1^2 + a_1\omega_2^2) \end{pmatrix}.$$

Clearly, the largest invariant set is constituted by the line of equation $\omega_1 = 0$ in this plane. Now the trajectories of the closed-loop system (3) evolve on the cylinders of equation $-a_2\omega_1^2 + a_1\omega_2^2 = c$ (the function $\omega \mapsto -a_2\omega_1^2 + a_1\omega_2^2$ is a first integral for system (1)) so we can say that each trajectory tends towards the set $\{A_c, B_c\}$ where A_c and B_c are the points of coordinates $\left(0, \sqrt{\frac{c}{a_1}}, 0\right)$ and $\left(0, -\sqrt{\frac{c}{a_1}}, 0\right)$. More precisely, we have:

PROPOSITION 2. *For the closed-loop system (3) restricted to the cylinder of equation* $-a_2\omega_1^2 + a_1\omega_2^2 = c$ $(c > 0)$, *we have the following description:*

- *the point A_c is an equilibrium point which is locally asymptotically stable,*
- *there is 2 trajectories whose ω-limit set is constituted by point B_c,*
- *the ω-limit set of the points which are not on these trajectories is equal to point A_c.*

PROOF. The equations of system (3) when restricted to the cylinder are:

$$\begin{aligned}\dot{\theta} &= -\sqrt{-a_1a_2}\,\omega_3,\\ \dot{\omega}_3 &= -\omega_3 + r^3\sqrt{-a_1a_2}\cos\theta.\end{aligned} \tag{4}$$

Around point A_c, the matrix of the linearized system is:

$$\begin{pmatrix} 0 & -\sqrt{-a_1a_2} \\ -r^3\sqrt{-a_1a_2} & -1 \end{pmatrix},$$

which is asymptotically stable. This proves the first point of the proposition.

Around point B_c, the matrix of the linearized system is:

$$\begin{pmatrix} 0 & -\sqrt{-a_1a_2} \\ r^3\sqrt{-a_1a_2} & -1 \end{pmatrix},$$

which has two eigenvalues of unlike signs. Locally, point B_c is a saddle-point, this proves the second point of the proposition.

As explained above all the trajectories of system (4) tend towards the set $\{A_c, B_c\}$. Let us denote by X the vector field for the system (4), by $X_t(.)$ the flow generated by this vector field and by d the distance on the cylinder. For a point M on the cylinder, if we have

$$\forall \varepsilon > 0, \qquad \exists T > 0 : d(X_T(M), B_c) < \varepsilon, \tag{5}$$

then, since point B_c is locally a saddle-point, M is on one of the two trajectories whose ω-limit set is B_c (or is equal to point B_c itself). If (5) is not true, then

$$\forall \varepsilon > 0, \qquad \exists T > 0 : d(X_T(M), A_c) < \varepsilon, \tag{6}$$

and if ε is choosen sufficiently small, $X_T(M)$ lies in the basin of attraction of point A_c which proves that the ω-limit set of M is A_c. ∎

Thus system (4), when restricted to an dense open subset of the cylinder (the cylinder minus point B_c and the two trajectories of Proposition 2), is globally asymptotically stable about point A_c.

REMARK. A slight modification in the feedback u (itemization suffices to put $u(\omega) = -a_3\omega_1\omega_2 - \omega_3 + (-\alpha a_2\omega_1 + \beta a_1\omega_2)(-a_2\omega_1^2 + a_1\omega_2^2))$ can transform points A_c and B_c into any pair of opposite points in the plane of equation $\omega_3 = 0$.

From the point of view of the stabilization about the middle axis, is itemization possible to get the best design for the trajectories of the closed-loop system (1) on the cylinder? If we use analytic feedbacks, we cannot have only one singular point in the intersection of the cylinder and the plane $\omega_3 = 0$, but can we have two singular points, say A and B, such that A is locally asymptoticaly stable and such that the α-limit set of every point which is different from B on the cylinder is A and the ω-limit set of every point which is different from A is B? As stated in the following proposition, the answer is no.

PROPOSITION 3. *Let X be a vector field defined on a finite dimensional manifold $\mathcal{M}$ and suppose that there exist two singular points A and Ω for X such that:*

- *Ω is locally asymptotically stable,*
- *the ω-limit set of every point in $\mathcal{M}$ which is different from A is Ω,*
- *the α-limit set of every point in $\mathcal{M}$ which is different from Ω is A.*

Then $\mathcal{M}$ is a compact manifold.

PROOF. Let $\mathbf{x} = (x_n)_{n\geq 0}$ be a sequence of points in $\mathcal{M}$, we will prove that we can extract from $\boldsymbol{x}$ a convergent subsequence.

Let U and V be two open disjoint neighborhoods of A and Ω, respectively. If $x_n \in U$ or $x_n \in V$ for an infinity of indices n, then we can extract a convergent subsequence from $\boldsymbol{x}$ because both $\bar{U}$ and $\bar{V}$ are compact. Now suppose that $x_n \in U \cup V$ only for a finite number of indices n, even if we have to suppress some terms in the sequence $\boldsymbol{x}$, we can assume that $x_n \notin U \cup V$ for all n.

Since Ω is stable, there exist an open neighborhood $V_0 \subset V$ of Ω such that for every initial condition $z \in V_0$, $X_t(z) \in V$ for all $t \geq 0$. Since A is the α-limit set of every point in $\mathcal{M}$, for all n there exist $t_n \geq 0$ such that $X_{-t_n}(x_n)\partial U$ and since $\bar{U}$ is compact, there exist a subsequence $\left(X_{-t_{n_k}}(x_{n_k})\right)_{k\geq 0}$ of $(X_{-t_n}(x_n))_{n\geq 0}$ which converges towards $y \in \partial U$.

If the sequence $(t_{n_k})_{n_k\geq 0}$ is bounded, we can suppose (even if we have to consider a subsequence of $(t_{n_k})_{n_k\geq 0}$) that t_{n_k} tends to $T > 0$, and then we have

$$\lim_{n_k\to+\infty} X_{t_{n_k}}\left(X_{-t_{n_k}}(x_{n_k})\right) = \lim_{n_k\to+\infty} x_{n_k} = X_T(y).$$

If the sequence $(t_{n_k})_{n_k\geq 0}$ is unbounded, consider a positive number T such that $X_T(y) \in V_0$ (such a number exists since the ω-limit set of y is Ω). For all t_{n_k} sufficiently large, we have $X_T\left(X_{-t_{n_k}}(x_{n_k})\right) \in V_0$ because $\lim X_{-t_{n_k}}(x_{n_k}) = y$, now we have $X_{t_{n_k}-T}\left(X_{-t_{n_k}}(x_{n_k})\right) = x_{n_k}$ with $t_{n_k} - T > 0$ if t_{n_k} is large enough but this is impossible because $X_t(z) \in V$ for all $t \geq 0$ and all $z \in V_0$. ∎

REMARK. There is a major topological difference between our problem and the one considered by Aeyels and Szafranski [7]: in their case, the invariant manifolds are composed with two connected components diffeomorphic to $\mathbb{R}^2$. In our case, the invariant manifolds are cylinders which are connected and, as explained above, we cannot avoid to have two singular points on the cylinder.

REFERENCES

1. A.M. Bloch and J.E. Marsden., Stabilization of rigid body dynamics by the Energy-Casimir method, *Systems & Control Letters* **14**, 341–346 (1990).
2. D. Aeyels, Stabilization by smooth feedback of the angular velocity of a rigid body, *Systems & Control Letters* **5**, 59–63 (1985).
3. E. Sontag and H.J. Sussmann, Further comments on the stabilizability of the angular velocity of the rigid body, *Systems & Control Letters* **12**, 213–217 (1988).
4. R. Outbib and G. Sallet, Stabilizability of the angular velocity of a rigid body revisited, *Systems & Control Letters* **18**, 93–98 (1992).
5. R.W. Brockett, Asymptotic stability and feedback stabilization, *Differential Geometric Control Theory*, (Edited by R.W. Brockett, Milmann, and H.J. Sussmann), pp. 181–191, (1983).
6. J. Lasalle and S. Lefschetz, *Stability by Liapunov's Direct Method with Applications*, Academic Press, New York, (1961).
7. D. Aeyels and M. Szafranski, Comments on the stabilizability of angular velocity of a rigid body, *Systems & Control Letters* **1**, 35–39 (1988).